

OPTIMAL STOPPING AND STOCHASTIC CONTROL DIFFERENTIAL GAMES FOR JUMP DIFFUSIONS

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1 Introduction

Let $X(t) = X(t, \omega) \in [0, \infty) \times \Omega$ be a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ representing the wealth of an investment at time t . The owner of the investment wants to find the optimal time for selling the investment. If we interpret “optimal” in the sense of “risk minimal”, then the problem is to find a stopping time $\tau = \tau(\omega)$ which minimizes $\rho(X(\tau))$, where ρ denotes a *risk measure*. If the risk measure ρ is chosen to be a convex risk measure in the sense of [5] and (or) [4], then it can be given the representation

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{N}} \{ \mathbb{E}_{\mathbb{Q}}[-X] - \zeta(Q) \}, \quad (1)$$

for some set \mathcal{N} of probability measures $\mathbb{Q} \ll \mathbb{P}$ and some convex “penalty” function $\zeta : \mathcal{N} \rightarrow \mathbb{R}$.

Using this representation the optimal stopping problem above gets the form

$$\inf_{\tau \in \mathcal{T}} \left(\sup_{\mathbb{Q} \in \mathcal{N}} \{ \mathbb{E}_{\mathbb{Q}}[-X(\tau)] - \zeta(Q) \} \right) \quad (2)$$

where \mathcal{T} is a given family of admissible \mathcal{F}_t -stopping times. This may be regarded as an *optimal stopping-stochastic control differential game*.

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In this paper we study this problem in a jump diffusion context. In Section 2 we formulate a general optimal stopping-stochastic control differential game problem in this context and we prove a general verification theorem for such games in terms of variational inequality-Hamilton-Jacobi-Bellman (VIHJB) equations. In Section 3 we show that the value function of the game in Section 2 is the unique viscosity solution of the associated VIHJB equations. Finally, in Section 4 we apply the general results obtained in Section 2 and 3 to study problem (2). By parametrizing the measures $\mathbb{Q} \in \mathcal{N}$ by a stochastic process $\theta(t, z) = (\theta_0(t), \theta_1(t, z))$ we may regard (2) as a special case of the general game in Section 2. We use this to solve the problem in some special cases.

2 General formulation

In this section we put the problem in the introduction into a general framework of optimal stopping and stochastic control differential game for jump diffusions and we prove a verification theorem for the value function of such a game. We refer to [9] for information about optimal stopping and stochastic control for jump diffusions. The following presentation follows [8] closely.

Suppose the state $Y(t) = Y^u(t) = Y_t^{y,u}$ at time t is given as the solution of a stochastic differential equation of the form

$$\begin{cases} dY(t) = b(Y(t), u_0(t)) dt + \sigma(Y(t), u_0(t)) dB(t) \\ + \int_{\mathbb{R}_0^k} \gamma(Y(t^-), u_1(t, z), z) \tilde{N}(dt, dz); \\ Y(0) = y \in \mathbb{R}^k. \end{cases} \quad (3)$$

Here $b : \mathbb{R}^k \times K \rightarrow \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times K \rightarrow \mathbb{R}^{k \times k}$ and $\gamma : \mathbb{R}^k \times K \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times k}$ are given functions, $B(t)$ is a k -dimensional Brownian motion and $\tilde{N}(\cdot, \cdot) = (\tilde{N}_1(\cdot, \cdot), \dots, \tilde{N}_k(\cdot, \cdot))$ are k independent compensated Poisson random measures independent of $B(\cdot)$, while K is a given subset of \mathbb{R}^p .

We may regard $u(t, z) = (u_0(t), u_1(t, z))$ as our control process, assumed to be càdlàg, \mathcal{F}_t -adapted and with values in $K \times K$ for a.a. t, z, ω .

Thus $Y(t) = Y^{(u)}(t)$ is a *controlled jump diffusion*.

Let $f : \mathbb{R}^k \times K \rightarrow \mathbb{R}$ and $g : \mathbb{R}^k \rightarrow \mathbb{R}$ be given functions. Let \mathcal{A} be a given set of controls contained in the set of $u = (u_0, u_1)$ such that (3) has a unique strong solution and such that

$$\mathbb{E}^y \left[\int_0^{\tau_S} |f(Y(t), u(t))| dt \right] < \infty \quad (4)$$

(where \mathbb{E}^y denotes expectation when $Y(0) = y$) where

$$\tau_S = \inf \{t > 0; Y(t) \notin \mathcal{S}\} \quad (\text{the bankruptcy time}) \quad (5)$$

is the first exit time of a given open *solvency set* $\mathcal{S} \subset \mathbb{R}^k$. We let \mathcal{T} denote the set of all stopping times $\tau \leq \tau_S$. We assume that

$$\{g^-(X(\tau))\}_{\tau \in \mathcal{T}} \quad \text{is uniformly integrable.} \quad (6)$$

For $\tau \in \mathcal{T}$ and $u \in \mathcal{A}$ we define the *performance functional* $J^{\tau,u}(y)$ by

$$J^{\tau,u}(y) = \mathbb{E}^y \left[\int_0^\tau f(Y(t), u(t)) dt + g(Y(\tau)) \right] \quad (7)$$

(we interpret $g(Y(\tau))$ as 0 if $\tau = \infty$).

We regard τ as the “control” of player number 1 and u as the control of player number 2, and consider the *stochastic differential game* to find the value function Φ and an optimal pair $(\tau^*, u^*) \in \mathcal{T} \times \mathcal{A}$ such that

$$\Phi(y) = \inf_{u \in \mathcal{A}} \left(\sup_{\tau \in \mathcal{T}} J^{\tau,u}(y) \right) = J^{\tau^*, u^*}(y). \quad (8)$$

We restrict ourselves to Markov controls $u = (u_0, u_1)$, i.e. we assume that $u_0(t) = \bar{u}_0(Y(t))$ and $u_1(t) = \bar{u}_1(Y(t), z)$ for some functions $\bar{u}_0 : \mathbb{R}^k \rightarrow K$, $\bar{u}_1 : \mathbb{R}^k \times \mathbb{R}^k \rightarrow K$. For simplicity of notation we will in the following not distinguish between u_0 and \bar{u}_0 , u_1 and \bar{u}_1 .

When the control u is Markovian the corresponding process $Y^{(u)}(t)$ becomes a Markov process, with generator A^u given by

$$\begin{aligned} A^u \varphi(y) &= \sum_{i=1}^k b_i(y, u_0(y)) \frac{\partial \varphi}{\partial y_i}(y) \\ &+ \frac{1}{2} \sum_{i,j=1}^k (\sigma \sigma^t)_{ij}(y, u_0(y)) \frac{\partial^2 \varphi}{\partial y_i \partial y_j}(y) \\ &+ \sum_{j=1}^k \int_{\mathbb{R}} \{ \varphi(y + \gamma^{(j)}(y, u_1(y, z), z)) - \varphi(y) \\ &\quad - \nabla \varphi(y) \cdot \gamma^{(j)}(y, u_1(y, z), z) \} \nu_j(dz); \quad \varphi \in \mathcal{C}^2(\mathbb{R}^k). \end{aligned} \quad (9)$$

Here $\nabla \varphi = (\frac{\partial \varphi}{\partial y_1}, \dots, \frac{\partial \varphi}{\partial y_k})$ is the gradient of φ and $\gamma^{(j)}$ is column number j of the $k \times k$ matrix γ .

We can now formulate the main result of this section:

Theorem 2.1 (*Verification theorem for stopping-control games*)

Suppose there exists a function $\varphi : \bar{\mathcal{S}} \rightarrow \mathbb{R}$ such that

(i) $\varphi \in \mathcal{C}^1(\mathcal{S}) \cap \mathcal{C}(\bar{\mathcal{S}})$

(ii) $\varphi \geq g$ on \mathcal{S}

Define

$$D = \{y \in \mathcal{S}; \varphi(y) > g(y)\} \quad (\text{the continuation region}). \quad (10)$$

Suppose, with $Y(t) = Y^{(u)}(t)$,

(iii) $\mathbb{E}^y \left[\int_0^{\tau_S} \chi_{\partial D}(Y(t)) dt \right] = 0$ for all $u \in \mathcal{A}$

(iv) ∂D is a Lipschitz surface

(v) $\varphi \in \mathcal{C}^2(\mathcal{S} \setminus \partial D)$, with locally bounded derivatives near ∂D

(vi) there exists $\hat{u} \in \mathcal{A}$ such that

$$A^{\hat{u}} \varphi(y) + f(y, \hat{u}(y)) = \inf_{u \in \mathcal{A}} \{ A^u \varphi(y) + f(y, u(y)) \} \begin{cases} = 0, & \text{for } y \in D, \\ \leq 0, & \text{for } y \in \mathcal{S} \setminus D. \end{cases}$$

(vii) $\mathbb{E}^y [|\varphi(Y(\tau))| + \int_0^\tau |A^u \varphi(Y(t))| dt] < \infty$, for all $\tau \in \mathcal{T}$ and all $u \in \mathcal{A}$.

For $u \in \mathcal{A}$ define

$$\tau_D = \tau_D^{(u)} = \inf\{t > 0; Y^{(u)}(t) \notin D\} \quad (11)$$

and, in particular,

$$\hat{\tau} = \tau_D^{(\hat{u})} = \inf\{t > 0; Y^{(\hat{u})}(t) \notin D\}.$$

(viii) Suppose that the family $\{\varphi(Y(\tau)); \tau \in \mathcal{T}, \tau \leq \tau_D\}$ is uniformly integrable, for each $u \in \mathcal{A}$, $y \in \mathcal{S}$.

Then $\varphi(y) = \Phi(y)$ and $(\hat{\tau}, \hat{u}) \in \mathcal{T} \times \mathcal{A}$ is an optimal pair, in the sense that

$$\Phi(y) = \inf_u \left(\sup_\tau J^{\tau, u}(y) \right) = \sup_\tau J^{\tau, \hat{u}}(y) = J^{\hat{\tau}, \hat{u}}(y) = \varphi(y) = \inf_u J^{\tau_D, u}(y) = \sup_\tau \left(\inf_u J^{\tau, u}(y) \right). \quad (12)$$

Proof. Choose $\tau \in \mathcal{T}$ and let $\hat{u} \in \mathcal{A}$ be as in (vi). By an approximation argument (see Theorem 3.1 in [9]) we may assume that $\varphi \in \mathcal{C}^2(\mathcal{S})$. Then by the Dynkin formula (see Theorem 1.24 in [9]) and (vi) we have, with $\hat{Y} = Y^{(\hat{u})}$

$$\begin{aligned} \mathbb{E}^y \left[\varphi \left(\hat{Y}(\tau_m) \right) \right] &= \varphi(y) + \mathbb{E}^y \left[\int_0^{\tau_m} A^{\hat{u}} \varphi \left(\hat{Y}(t) \right) dt \right] \\ &\leq \varphi(y) - \mathbb{E}^y \left[\int_0^{\tau_m} f \left(\hat{Y}(t), \hat{u}(t) \right) dt \right], \end{aligned}$$

where $\tau_m = \tau \wedge m$; $m = 1, 2, \dots$.

Letting $m \rightarrow \infty$ this gives, by (4), (6), (vii), (i) and the Fatou Lemma,

$$\begin{aligned} \varphi(y) &\geq \liminf_{m \rightarrow \infty} \mathbb{E}^y \left[\int_0^{\tau_m} f \left(\hat{Y}(t), \hat{u}(t) \right) dt + \varphi(\hat{Y}(\tau_m)) \right] \\ &\geq \mathbb{E}^y \left[\int_0^\tau f \left(\hat{Y}(t), \hat{u}(t) \right) dt + g(\hat{Y}(\tau)) \chi_{\{\tau < \infty\}} \right] = J^{\tau, \hat{u}}(y). \end{aligned} \quad (13)$$

Since this holds for all τ we have

$$\varphi(y) \geq \sup_\tau J^{\tau, \hat{u}}(y) \geq \inf_u \left(\sup_\tau J^{\tau, u}(y) \right), \text{ for all } u \in \mathcal{A}. \quad (14)$$

Next, for given $u \in \mathcal{A}$ define, with $Y(t) = Y^{(u)}(t)$,

$$\tau_D = \tau_D^u = \inf\{t > 0; Y(t) \notin D\}.$$

Choose a sequence $\{D_m\}_{m=1}^\infty$ of open sets such that \bar{D}_m is compact, $\bar{D}_m \subset D_{m+1}$ and $D = \bigcup_{m=1}^\infty D_m$

and define

$$\tau_D(m) = m \wedge \inf\{t > 0; Y(t) \notin D_m\}.$$

By the Dynkin formula we have, by (vi), for $m = 1, 2, \dots$,

$$\begin{aligned}\varphi(y) &= \mathbb{E}^y \left[- \int_0^{\tau_D(m)} A^u \varphi(Y(t)) dt + \varphi(Y(\tau_D(m))) \right] \\ &\leq \mathbb{E}^y \left[\int_0^{\tau_D(m)} f(Y(t), u(t)) dt + \varphi(Y(\tau_D(m))) \right].\end{aligned}\tag{15}$$

By the quasi-left continuity of $Y(\cdot)$ (see [6], Proposition I. 2. 26 and Proposition I. 3. 27), we get

$$Y(\tau_D(m)) \rightarrow Y(\tau_D) \text{ a.s. as } m \rightarrow \infty.$$

Therefore, if we let $m \rightarrow \infty$ in (15) we get

$$\varphi(y) \leq \mathbb{E}^y \left[\int_0^{\tau_D} f(Y(t), u(t)) dt + g(Y(\tau_D)) \right] = J^{\tau_D, u}(y).$$

Since this holds for all $u \in \mathcal{A}$ we get

$$\varphi(y) \leq \inf_u J^{\tau_D, u}(y) \leq \sup_\tau \left(\inf_u J^{\tau, u}(y) \right).\tag{16}$$

In particular, applying this to $u = \hat{u}$ we get equality, i.e.

$$\varphi(y) = J^{\hat{\tau}, \hat{u}}(y).\tag{17}$$

Combining (14), (16) and (17) we obtain

$$\begin{aligned}\inf_u \left(\sup_\tau J^{\tau, u}(y) \right) &\leq \sup_\tau J^{\tau, \hat{u}}(y) \leq \varphi(y) = J^{\hat{\tau}, \hat{u}}(y) = \varphi(y) \\ &\leq \inf_u J^{\tau_D, u}(y) \leq \sup_\tau \left(\inf_u J^{\tau, u}(y) \right) \leq \inf_u \left(\sup_\tau J^{\tau, u}(y) \right).\end{aligned}\tag{18}$$

Since we always have

$$\sup_\tau \left(\inf_u J^{\tau, u}(y) \right) \leq \inf_u \left(\sup_\tau J^{\tau, u}(y) \right)\tag{19}$$

we conclude that we have equality everywhere in (18) and the proof is complete. ■

3 Viscosity solutions

Let the state, $Y(t) = Y^u(t)$, be given by equation (3), the performance functional by equation (7) and the value function by equation (8). In the following we will assume that the functions b, σ, γ, f, g are continuous with respect to (y, u) . Further, the following standard assumptions are adopted; there exists $C > 0$, $\alpha : \mathbb{R}^k \rightarrow \mathbb{R}^k$ with $\int \alpha^2(z) \nu(dz) < \infty$ such that for all $x, y \in \mathbb{R}^k, z \in \mathbb{R}^k$ and $u \in K$,

$$\text{A1. } |b(x, u) - b(y, u)| + |\sigma(x, u) - \sigma(y, u)| \leq C|x - y|,$$

$$\text{A2. } |f(x, u) - f(y, u)| + |g(x, u) - g(y, u)| \leq C|x - y|,$$

$$\text{A3. } |\gamma(x, u_1, z) - \gamma(y, u_1, z)| \leq \alpha(z)|x - y|,$$

$$\text{A4. } |\gamma(x, u_1, z)| \leq \alpha(z)(1 + |x|) \text{ and } |\gamma(x, u_1, z)|\mathbf{1}_{|z|<1} \leq C_x, C_x \in \mathbb{R}.$$

Let us define a HJB variational inequality by

$$\max \left\{ \inf_{u \in K} [A^u \varphi(y) + f(y, u(y))], g(y) - \varphi(y) \right\} = 0, \quad (20)$$

and

$$\varphi = g \text{ on } \partial \mathcal{S}. \quad (21)$$

where $A^y \varphi(y)$ is defined by equation (9).

Definition 3.1 (Viscosity solutions) A locally bounded function $\varphi \in USC(\bar{\mathcal{S}})$ is called a viscosity subsolution of (20)-(21) in \mathcal{S} if (21) holds and for each $\psi \in C_0^2(\mathcal{S})$ and each $y_0 \in \mathcal{S}$ such that $\psi \geq \varphi$ on \mathcal{S} and $\psi(y_0) = \varphi(y_0)$, we have

$$\max \left\{ \inf_{u \in K} [A^u \psi(y_0) + f(y_0, u(y_0))], g(y_0) - \psi(y_0) \right\} \geq 0 \quad (22)$$

A function $\varphi \in LSC(\bar{\mathcal{S}})$ is called a viscosity supersolution of the (20)-(21) in \mathcal{S} if (21) holds and for each $\psi \in C_0^2(\mathcal{S})$ and each $y_0 \in \mathcal{S}$ such that $\psi \leq \varphi$ on \mathcal{S} and $\psi(y_0) = \varphi(y_0)$, we have

$$\max \left\{ \inf_{u \in K} [A^u \psi(y_0) + f(y_0, u(y_0))], g(y_0) - \psi(y_0) \right\} \leq 0 \quad (23)$$

Further, if $\varphi \in C([0, T] \times \mathbb{R}^n)$ is both a viscosity subsolution and a viscosity supersolution it is called a viscosity solution.

Proposition 3.2 (Bellman's principle of optimality) The stochastic control version of Bellman's principle of optimality:

Let Φ be as in (8). Then we have

$$(i) \quad \forall h > 0, \forall y \in \mathbb{R}^k$$

$$\Phi(y) = \sup_{\tau \in \mathbb{T}} \inf_{u \in A} E^y \left[\int_0^{\tau \wedge h} f(Y(s), u(s)) ds + g(Y_\tau) \mathbf{1}_{\tau < h} + \psi(Y_h) \mathbf{1}_{h \leq \tau} \right].$$

(ii) Let $\varepsilon > 0$, $y \in \mathbb{R}^k$, $u \in K$ and define the stopping time

$$\tau_{y,u}^\varepsilon = \inf \{ 0 \leq t \leq \tau_s; \Phi(Y_t^{y,u}) \geq g(Y_t^{y,u}) - \varepsilon \}.$$

Then, if $\tau_u \leq \tau_{y,u}^\varepsilon$ for all $u \in A$, we have that:

$$\Phi(y) = \inf_{u \in A} E^y \left[\int_0^{\tau_u} f(Y(s)) ds + g(Y_{\tau_u}) \right].$$

Theorem 3.3 *Under assumptions A1-A4, the value function Φ is a viscosity solution of (20)-(21).*

Proof. Φ is continuous according to the estimates of the moments of the jump diffusion state process (see Lemma 3.1 p.9 in [11]) and from Lipschitz condition A2 on f and g we get that

$$\Phi(y) = g(y) \text{ on } \partial\mathcal{S}.$$

We now prove that Φ is a subsolution of (20)-(21). Let $\psi \in C_0^2(\mathcal{S})$ and $y_0 \in \mathcal{S}$ such that

$$0 = (\psi - \Phi)(y_0) = \min_y (\psi - \Phi). \quad (24)$$

Define

$$D = \{y \in \mathcal{S} | \Phi(y) > g(y)\}.$$

If $y_0 \notin D$ then $g(y_0) = \Phi(y_0)$ and hence (22) holds. Next suppose $y_0 \in D$. Then we have by Proposition 3.2 for $\hat{\tau} = \tau_D$ and $h > 0$ small enough:

$$\Phi(y_0) = \inf_{u(\cdot) \in A} E^{y_0} \left[\int_0^h f(Y^{y_0}(t), u(t)) dt + \Phi(Y^{y_0}(h)) \right].$$

From (24) we get

$$0 \leq \inf_{u(\cdot) \in A} E^{y_0} \left[\int_0^h f(Y^{y_0}(t), u(t)) dt + \psi(Y^{y_0}(h)) - \psi(y_0) \right].$$

By Itô's formula we obtain that

$$0 \leq \inf_{u(\cdot) \in A} \frac{1}{h} E^{y_0} \left[\int_0^h [A^u \psi(Y_t^{y_0}) + f(Y^{y_0}(t), u(t))] dt \right].$$

Using assumptions A1-A4 with estimates on the moments of a jump diffusion and by letting $h \rightarrow 0^+$, we have

$$\inf_{u \in K} [A^u \psi(y_0) + f(y_0, u(y_0))] \geq 0,$$

and hence

$$\max \left\{ \inf_{u \in K} [A^u \psi(y_0) + f(y_0, u(y_0))], g(y_0) - \psi(y_0) \right\} \geq 0.$$

This shows that Φ is a viscosity subsolution. The proof for supersolution is similar. ■

The problem of showing uniqueness of viscosity solution is not addressed in this paper but will be considered in a future article.

4 Examples

Let us look at some control problems where we include stopping times as one of the controls. We then apply the result of the previous section to find a solution. We will look at both a jump and a non-jump market.

Example 4.1 (Optimal Resource Extraction in a Worst Case Scenario) *Let*

$$dP(t) = P(t)[\alpha dt + \beta dB(t) + \int_{\mathbb{R}_0} \gamma(z) \tilde{N}(dt, dz)]; P(0) = y_1 > 0,$$

where α, β are constants and $\gamma(z)$ is a given function such that $\int_{\mathbb{R}_0} \gamma^2(z) \nu(dz) < \infty$. Let $Q(t)$ be the amount of remaining resources at time t , and let the dynamics be described by

$$dQ(t) = -u(t)Q(t)dt; Q(0) = y_2 \geq 0.$$

where $u(t)$ controls the consumption rate of the resource $Q(t)$, and m is the maximum extraction rate.

We let

$$dY(t) = \begin{cases} dY_0(t) &= dt \\ dY_1(t) &= dP(t); P(0) = y_1 > 0, \\ dY_2(t) &= dQ(t); Q(0) = y_2 \geq 0, \\ dY_3(t) &= -Y_3(t) \left[\theta_0(t)dB(t) + \int_{\mathbb{R}_0} \theta_1(t, z) \tilde{N}(dt, dz) \right]; Y_3(0) = y_3 > 0. \end{cases} \quad (25)$$

Let the running cost be given by $K_0 + K_1 u_t$ ($K_0, K_1 \geq 0$, constants). Then we let our performance functional be given by, with $\theta = (\theta_0, \theta_1)$,

$$\begin{aligned} J^{\tau, u, \theta}(s, y_1, y_2, y_3) \\ = E^y \left[\int_0^\tau e^{-\delta(s+t)} (u(t)(P(t)Q(t) - K_1) - K_0) Y_3(t) dt + e^{-\delta(s+\tau)} (MP(\tau)Q(\tau) - a) Y_3(\tau) \right], \end{aligned} \quad (26)$$

where $\delta > 0$ is the discounting rate and $M > 0, a > 0$ are constants (a can be seen as a transaction cost). Our problem is to find $(\hat{\tau}, \hat{u}, \hat{\theta})$ in $\mathcal{T} \times U \times \Theta$ such that

$$\Phi(y) = \Phi(s, y_1, y_2, y_3) = \sup_u \left[\inf_\theta \left(\sup_\tau J^{\tau, u, \theta}(y) \right) \right] = J^{\hat{\tau}, \hat{u}, \hat{\theta}}(y). \quad (27)$$

Then the generator of $Y^{u, \theta}$ is given by;

$$\begin{aligned} A^{u, \theta} \varphi(y) &= A^{u, \theta} \varphi(s, y_1, y_2, y_3) = \frac{\partial \varphi}{\partial s} + y_1 \alpha \frac{\partial \varphi}{\partial y_1} - u y_2 \frac{\partial \varphi}{\partial y_2} + \frac{1}{2} y_1^2 \beta^2 \frac{\partial^2 \varphi}{\partial^2 y_1} + \frac{1}{2} y_3^2 \theta_0^2 \frac{\partial^2 \varphi}{\partial^2 y_3} \\ &\quad - y_1 y_3 \beta \theta_0 \frac{\partial^2 \varphi}{\partial y_1 \partial y_3} \\ &\quad + \int_{\mathbb{R}_0} \left\{ \varphi(s, y_1 + y_1 \gamma(z), y_2, y_3 - y_3 \theta_1(z)) - \varphi(s, y_1, y_2, y_3) - y_1 \gamma(z) \frac{\partial \varphi}{\partial y_1} + y_3 \theta_1(z) \frac{\partial \varphi}{\partial y_3} \right\} \nu(dz). \end{aligned}$$

We need to find a subset D of $\mathcal{S} = \mathbb{R}_+^4 = [0, \infty)^4$ and $\varphi(s, y_1, y_2, y_3)$ such that

$$\varphi(s, y_1, y_2, y_3) = g(s, y_1, y_2, y_3) := e^{-\delta s} (M y_1 y_2 - a) y_3, \quad \forall (s, y_1, y_2, y_3) \notin D,$$

$$\varphi(s, y_1, y_2, y_3) \geq e^{-\delta s} (M y_1 y_2 - a) y_3, \quad \forall (s, y_1, y_2, y_3) \in \mathcal{S},$$

$$\begin{aligned} A^{u, \theta} \varphi(s, y_1, y_2, y_3) + f(s, y_1, y_2, y_3, u) &:= A^{u, \theta} \varphi(s, y_1, y_2, y_3) + e^{-\delta s} (u(y_1 y_2 - K_1) - K_0) y_3 \\ &\leq 0, \quad \forall (s, y_1, y_2, y_3) \in \mathcal{S} \setminus D, \quad \forall u \in [0, m], \end{aligned}$$

$$\sup_u \left[\inf_\theta \{ A^{u, \theta} \varphi(s, y_1, y_2, y_3) + e^{-\delta s} (u(y_1 y_2 - K_1) - K_0) y_3 \} \right] = 0, \quad \forall (s, y_1, y_2, y_3) \in D.$$

Then

$$\hat{\theta}_0 = \frac{y_1}{y_3} \beta \frac{\varphi_{13}}{\varphi_{33}}, \quad (28)$$

is a minimizer of $\theta_0 \mapsto A^{u,\theta} \varphi(s, y_1, y_2, y_3)$ where we are using the notation

$$\varphi_{ij} = \frac{\partial^2 \varphi}{\partial y_j \partial y_i}.$$

Let $\hat{\theta}_1(z)$ be the minimizer of $\theta_1(z) \mapsto A^{u,\theta} \varphi(y)$ and let \hat{u} be the maximizer of $u \mapsto A^{u,\theta} \varphi(y) + f(y, u)$ i.e.

$$u \mapsto A^{u,\theta} \varphi(s, y_1, y_2, y_3) + e^{-\delta s} u y_3 (y_1 y_2 - K_1) - u y_2 \varphi_2 - y_3 K_0. \quad (29)$$

Let us try a function on the form

$$\varphi(s, y_1, y_2, y_3) = e^{-\delta s} F(w), \text{ where } w = y_1 y_2 y_3. \quad (30)$$

Then

$$\hat{u} = \begin{cases} m, & \text{if } w F'(w) < w - y_3 K_1 \\ 0, & \text{otherwise,} \end{cases} \quad (31)$$

and

$$\hat{\theta} = \beta \left(1 + \frac{F'(w)}{F''(w)w} \right). \quad (32)$$

Further, the first order condition for $\hat{\theta}_1(z)$ is

$$\int_{\mathbb{R}_0} \left\{ (1 + \gamma(z)) F'(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F'(w) \right\} \nu(dz) = 0. \quad (33)$$

For $w F'(w) < w - y_3 K_1$ we have

$$\begin{aligned} A^{\hat{u}, \hat{\theta}} e^{-\delta s} F(y_1, y_2, y_3) &= -\delta e^{-\delta s} F(w) + w e^{-\delta s} \alpha F'(w) - m w e^{-\delta s} F'(w) \\ &+ \frac{1}{2} w^2 \beta^2 F''(w) e^{-\delta s} + \frac{1}{2} w^2 \beta^2 F''(w) e^{-\delta s} \left(1 + \left(\frac{F'(w)}{F''(w)w} \right)^2 + \frac{2F'(w)}{F''(w)w} \right) \\ &- w \beta^2 e^{-\delta s} \left(F'(w) + \frac{(F'(w))^2}{F''(w)w} + w F''(w) + F'(w) \right) \\ &+ e^{-\delta s} \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w \gamma(z) F'(w) + \hat{\theta}_1(z) w F'(w) \right\} \nu(dz) \\ &= -\delta e^{-\delta s} F(w) + w e^{-\delta s} \alpha F'(w) - m w e^{-\delta s} F'(w) \\ &+ \beta^2 e^{-\delta s} \left(-\frac{(F'(w))^2}{2F''(w)} - w F'(w) \right) \\ &+ e^{-\delta s} \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w \gamma(z) F'(w) + \hat{\theta}_1(z) w F'(w) \right\} \nu(dz). \end{aligned} \quad (34)$$

We then need that if $wF'(w) < w - y_3K_1$, then

$$\begin{aligned} A^{\hat{u}, \hat{\theta}} F(w) + (m(y_1y_2 - K_1) - K_0)y_3 &= -\delta F(w) + w\alpha F'(w) - mwF'(w) \\ &- \beta^2 \left(\frac{(F'(w))^2}{2F''(w)} + wF'(w) \right) \\ &+ \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w\gamma(z)F'(w) + \hat{\theta}_1(z)wF'(w) \right\} \nu(dz) \\ &+ (m(y_1y_2 - K_1) - K_0)y_3 = 0. \end{aligned} \quad (35)$$

Similarly, if $wF'(w) \geq w - y_3K_1$, then $\hat{u} = 0$ and hence we must have

$$\begin{aligned} A^{\hat{u}, \hat{\theta}} F(w) - K_0y_3 &= -\delta F(w) + w\alpha F'(w) \\ &- \beta^2 \left(\frac{(F'(w))^2}{2F''(w)} + wF'(w) \right) \\ &+ \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w\gamma(z)F'(w) + \hat{\theta}_1(z)wF'(w) \right\} \nu(dz) \\ &- K_0y_3 = 0. \end{aligned} \quad (36)$$

The continuation region D gets the form

$$D = \{(s, y_1, y_2, y_3) : F(w) > (My_1y_2 - a)y_3\}$$

Therefore we get the requirement

$$F(w) = (My_1y_2 - a)y_3, \quad \forall (s, y_1, y_2, y_3) \notin D. \quad (37)$$

In light of this requirement and in order for φ to be on the form (30) we see that we need K_0, K_1 and a to be zero. Hence we let $K_0 = K_1 = a = 0$ from now on. Then we need that F satisfies the variational inequality

$$\max\{A_0^{\hat{u}, \hat{\theta}} F(w) + \tilde{m}w, Mw - F(w)\} = 0, \quad w > 0, \quad (38)$$

where

$$\begin{aligned} A_0^{\hat{u}, \hat{\theta}} F(w) &= -\delta F(w) + w\alpha F'(w) - \tilde{m}wF'(w) - \beta^2 \left(\frac{(F'(w))^2}{2F''(w)} + wF'(w) \right) \\ &+ \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w\gamma(z)F'(w) + \hat{\theta}_1(z)wF'(w) \right\} \nu(dz), \end{aligned} \quad (39)$$

with

$$\tilde{m} := m\chi_{(-\infty, 1)}(F'(w)). \quad (40)$$

The variational inequality (38) - (40) is hard to solve analytically, but it may be accessible by numerical methods.

Exemple 4.2 (Worst case scenario optimal control and stopping in a Lévy -market) *Let our dynamics be given by*

$$\begin{aligned} dY_0(t) &= dt; & Y_0(0) &= s \in \mathbb{R}. \\ dY_1(t) &= (Y_1(t)\alpha(t) - u(t))dt + Y_1(t)\beta dB(t) \\ &\quad + Y_1(t^-) \int_{\mathbb{R}} \gamma(s, z) \tilde{N}(ds, dz); & Y_1(0) &= y_1 > 0. \\ dY_2(t) &= -Y_2(t)\theta_0(t)dB(t) - Y_2(t) \int_{\mathbb{R}} \theta_1(s, z) \tilde{N}(ds, dz); & Y_2(0) &= y_2 > 0. \end{aligned}$$

Solve

$$\Phi(s, x) = \sup_u \left[\sup_{\tau} \left(\inf_{\theta_0, \theta_1} J^{\theta, u, \tau}(s, x) \right) \right]$$

where

$$J^{\theta, u, \tau}(s, x) = E^x \left[\int_0^{\tau} e^{-\delta(s+t)} \frac{u^\lambda}{\lambda} Y_2(t) dt \right]$$

The interpretation of this problem is the following:

$Y_1(T)$ represents the size of the population (e.g. fish) when a harvesting strategy $u(t)$ is applied to it. The process $Y_2(t)$ represents the Radon-Nikodym derivative of a measure Q with respect to P , i.e.

$$Y_2(t) = \frac{d(Q|\mathcal{F}_t)}{d(P|\mathcal{F}_t)} = E\left[\frac{dQ}{dP}|\mathcal{F}_t\right]; 0 \leq t \leq T.$$

This means that we can write

$$\begin{aligned} J^{\theta, y, \tau}(s, x) &= E^x \left[\int_0^{\tau} e^{-\delta(s+t)} \chi_{[0, \tau](t)} \frac{u^\lambda(t)}{\lambda} E\left[\frac{dQ}{dP}|\mathcal{F}_t\right] dt \right] \\ &= E_Q^x \left[\int_0^{\tau} e^{-\delta(s+t)} \frac{u^\lambda(t)}{\lambda} dt \right]. \end{aligned}$$

Hence $J^{\theta, y, \tau}$ represents the expected utility up to the stopping time τ , measured in terms of a scenario (probability measure Q) chosen by the market. Therefore our problem may be regarded as a worst case scenario optimal harvesting/stopping problem. Alternatively, the problem may be interpreted as a risk minimizing optimal stopping and control problem. To see this, we use the following representation of a given convex risk measure ρ :

$$\rho(F) = \sup_{Q \in \mathcal{P}} \{E_Q[-F] - \varsigma(Q)\}; F \in L^\infty(P),$$

where \mathcal{P} is the set of all measures Q above and $\varsigma : \mathcal{P} \rightarrow \mathbb{R}$ is a given convex “penalty” function. If $\varsigma = 0$ as above, the risk measure ρ is called coherent. See [1], [4] and [5].

In this case our generator becomes

$$\begin{aligned} A^{u, \theta} \varphi(s, y_1, y_2) &= \frac{\partial \varphi}{\partial s} + (y_1 \alpha - u) \frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \frac{\partial^2 \varphi}{\partial^2 y_1} \\ &\quad + \frac{1}{2} y_2^2 \theta_0^2 \frac{\partial^2 \varphi}{\partial^2 y_2} - y_1 y_2 \beta \theta_0 \frac{\partial^2 \varphi}{\partial y_1 \partial y_2} \\ &\quad + \int_{\mathbb{R}} \left[\varphi(s, y_1 + y_1 \gamma(s, z), y_2 - y_2 \theta_1(s, z)) - \varphi(s, y_1, y_2) - y_1 \gamma(s, z) \frac{\partial \varphi}{\partial y_1} + y_2 \theta_1(z) \frac{\partial \varphi}{\partial y_2} \right] \nu(dz). \end{aligned}$$

and hence

$$\begin{aligned}
A^{u,\theta}\varphi(s, y_1, y_2) + f(s, y_1, y_2) &= \frac{\partial\varphi}{\partial s} + (y_1\alpha - u)\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\frac{\partial^2\varphi}{\partial^2 y_1} \\
&+ \frac{1}{2}y_2^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_2} \\
&+ \int_{\mathbb{R}} \left[\varphi(s, y_1 + y_1\gamma(s, z), y_2 - y_2\theta_1(s, z)) - \varphi(s, y_1, y_2) - y_1\gamma(s, z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2} \right] \nu(dz) \\
&+ e^{-\delta s} \frac{u^\lambda}{\lambda} y_2.
\end{aligned}$$

Imposing the first-order condition we get the following equations for the optimal control processes $\hat{\theta}_0, \hat{\theta}_1$ and \hat{u} :

$$\hat{\theta}_0 = \frac{y_1}{y_2} \beta \frac{\varphi_{12}}{\varphi_{22}},$$

$$\int_{\mathbb{R}} \{ \varphi_2(s, y_1 + y_1\gamma(s, z), y_2 - y_2\hat{\theta}_1(s, z)) - \varphi_2(s, y_1, y_2) \} \nu(dz) = 0,$$

and

$$\hat{u} = \left(\frac{e^{\delta s} \varphi_1}{y_2} \right)^{\frac{1}{\lambda-1}},$$

where $\varphi_i = \frac{\partial\varphi}{\partial y_i}$; $i = 1, 2$. This gives

$$\begin{aligned}
A^{\hat{u},\hat{\theta}}\varphi(s, y_1, y_2) + f(s, y_1, y_2, \hat{u}) &= \frac{\partial\varphi}{\partial s} + (y_1\alpha - (e^{\delta s} \frac{\varphi_1}{y_2})^{\frac{1}{\lambda-1}}) \varphi_1 + \frac{1}{2}y_1^2\beta^2\varphi_{11} - \frac{1}{2}y_1^2\beta^2\frac{\varphi_{12}^2}{\varphi_{22}} \quad (41) \\
&+ \int_{\mathbb{R}} \left[\varphi(s, y_1 + y_1\gamma(s, z), y_2 - y_2\hat{\theta}_1(s, z)) - \varphi(s, y_1, y_2) - y_1\gamma(s, z)\frac{\partial\varphi}{\partial y_1} + y_2\hat{\theta}_1(z)\frac{\partial\varphi}{\partial y_2} \right] \nu(dz) \\
&+ e^{-\delta s} \frac{\left(\frac{\varphi_1 e^{\delta s}}{y_2} \right)^{\frac{\lambda}{\lambda-1}}}{\lambda} y_2.
\end{aligned}$$

Let us try a value function of the form

$$\varphi(s, y_1, y_2) = e^{-\delta s} y_1^\lambda F(y_2), \quad (42)$$

for some function F (to be determined). Then

$$\hat{\theta}_0 = \beta \frac{\lambda \beta F'(y_2)}{y_2 F''(y_2)}, \quad (43)$$

$$\int_{\mathbb{R}} \{ (1 + \gamma(s, z))^\gamma F'(y_2 - y_2\hat{\theta}_1(s, z)) - F'(y_2) \} \nu(dz) = 0, \quad (44)$$

and

$$\hat{u} = \left(\frac{F(y_2)\lambda}{y_2} \right)^{\frac{1}{\lambda-1}} y_1. \quad (45)$$

With $\hat{\theta}_1$ as in (44) put

$$\begin{aligned} A_0^{\hat{\theta}, \hat{u}} F(y_2) = & -\delta F(y_2) + \left(\alpha - \left(\frac{\lambda F(y_2)}{y_2}\right)^{\frac{1}{\lambda-1}}\right) \lambda F(y_2) + \frac{1}{2} \beta^2 \lambda (\lambda - 1) F(y_2) - \frac{1}{2} \beta^2 \frac{F'^2(y_2)}{F''(y_2)} \\ & + \frac{y_2}{\lambda} \left(\frac{\lambda F(y_2)}{y_2}\right)^{\frac{\lambda}{\lambda-1}} + \int_{\mathbb{R}} \left[(1 + \gamma(z))^\lambda F(y_2 - y_2 \hat{\theta}_1(z)) \right. \\ & \left. - F(y_2) - \gamma(z) \lambda F(y_2) + y_2 \hat{\theta}_1(z) F'(y_2) \right] \nu(dz). \end{aligned} \quad (46)$$

Thus we see that the problem reduces to the problem of solving a non-linear variational-integro inequality as follows:

Suppose there exists a process $\hat{\theta}_1(s, z)$ satisfying (44) and a C^1 -function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that if we put

$$D = \{y_2 > 0; F(y_2) > 0\}$$

then $F \in C^2(D)$ and

$$A_0^{\hat{\theta}, \hat{u}} F(y_2) = 0 \text{ for } y_2 \in D.$$

Then the function φ given by (42) is the value function of the problem. The optimal control process are as in (43) - (45) and an optimal stopping time is

$$\tau^* = \inf\{t > 0; Y_2(t) \notin D\}.$$

Exemple 4.3 (Risk minimizing optimal portfolio and stopping)

$$dY_0(t) = dt; \quad Y_0(0) = s \in \mathbb{R}. \quad (47)$$

$$dY_1(t) = Y_1(t)[(r + (\alpha - r)\pi(t))dt + \beta\pi(t)dB(t)]; \quad Y_1(0) = y_1 > 0. \quad (48)$$

$$dY_2(t) = -Y_2(t)\theta(t)dB(t); \quad Y_2(0) = y_2 > 0, \quad (49)$$

where r, α and $\beta > 0$ are constants. Solve

$$\Phi(s, x) = \sup_{\pi} \left[\sup_{\tau} \left(\inf_{\theta} J^{\pi, \theta, \tau} \right) \right] \quad (50)$$

where

$$J^{\pi, \theta, \tau}(s, x) = E^x \left[e^{-\delta\tau} \lambda Y_1(\tau) Y_2(\tau) \right], \quad (51)$$

where $0 < \lambda \leq 1$ and $(1 - \lambda)$ is a percentage transaction cost. The generator is

$$\begin{aligned} A^{\theta, \pi} \varphi(s, y_1, y_2) + f(s, y_1, y_2) = & \frac{\partial \varphi}{\partial s} + y_1(r + (\alpha - r)\pi) \frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \pi^2 \frac{\partial^2 \varphi}{\partial^2 y_1} \\ & + \frac{1}{2} y_2^2 \theta^2 \frac{\partial^2 \varphi}{\partial^2 y_2} - y_1 y_2 \beta \theta \pi \frac{\partial^2 \varphi}{\partial y_1 \partial y_2}. \end{aligned}$$

From the first order conditions we get that

$$\hat{\pi} = \frac{(\alpha - r) \varphi_1 \varphi_{22}}{y_1 \beta^2 (\varphi_{12}^2 - \varphi_{11} \varphi_{22})},$$

and

$$\hat{\theta} = \frac{(\alpha - r)\varphi_1\varphi_{12}}{\beta y_2(\varphi_{12}^2 - \varphi_{11}\varphi_{22})}.$$

Let us try to put

$$\varphi(s, y_1, y_2) = e^{-\delta s} \lambda y_1 y_2. \quad (52)$$

Then we get

$$A^{\hat{\theta}, \hat{\pi}} \varphi(s, y_1, y_2) = y_1 y_2 (r - \delta),$$

$$\hat{\theta} = \frac{\alpha - r}{\beta}. \quad (53)$$

and

$$\hat{\pi} = 0. \quad (54)$$

So if

$$r - \delta \leq 0,$$

then $A^{\hat{\theta}, \hat{\pi}} \varphi \leq 0$ and the best is to stop immediately and $\varphi = \Phi$. If

$$r - \delta > 0,$$

then

$$D = [0, T] \times \mathbb{R}^k \times \mathbb{R}^k,$$

so $\hat{\tau} = T$.

Remark 4.1 Note that the optimal value given in (53) for $\hat{\theta}$ corresponds to choosing the measure Q defined by

$$dQ(\omega) = Y_2(T) dP(\omega)$$

to be an equivalent martingale measure for the underlying financial market $(S_0(t), S_1(t))$ defined by

$$\begin{aligned} dS_0(t) &= r dt; S_0(0) = 0, \\ dS_1(t) &= S_1(t)[\alpha dt + \beta dB(t)]; S_1(0) > 0. \end{aligned}$$

This illustrates that equivalent martingale measures often appear as solutions of stochastic differential games between the agent and the market. This was first proved in [10] and subsequent in a partial information context in [2] and [3].

References

- [1] P. ARTZNER, F. DELBAEN, J. M. EBER AND D. HEATH: Coherent measures of risk. *Mathematical Finance* 4, 203-228 (1999).
- [2] T. T. K. AN AND B. ØKSENDAL: A maximum principle for stochastic differential games with partial information. *J. of Optim. Theory and Appl.* 139, 463-483 (2008).
- [3] T. T. K. AN, B. ØKSENDAL AND Y. Y. OKUR: A Malliavin calculus approach to general stochastic differential games with partial information. *Eprint, Univ. of Oslo* 26,(2008).
- [4] M. FRITTELLI AND E. GIANIN ROSAZZA : Putting order in risk measures. *J. Banking & Finance* 26, 1473-1486 (2002).
- [5] H. FÖLLMER AND A. SCHIED : Convex measures of risk and trading constraints. *Finance Stochast.* 6, 449-447 (2002).
- [6] J. JACOD AND A. SHIRYAEV : *Limit Theorems for Stochastic Processes*, Second Edition. Springer-Verlag Berlin (2003).
- [7] E.R. JAKOBSEN AND K.H. KARLSEN : A maximum principle for semi-continuous functions applicable to integro-partial differential Equations. *Nonlinear Differential Equations Appl.* 13, 1-29 (2006).
- [8] S. MATARAMVURA AND B. ØKSENDAL : Risk minimizing portfolios and HJBI equations for stochastic differential games. *Stochastics* 80, n°4, 317-337 (2008).
- [9] B. ØKSENDAL AND A. SULEM : *Applied Stochastic Control of Jump Diffusions*, Second Edition. Springer-Verlag Berlin (2007).
- [10] B. ØKSENDAL AND A. SULEM: A game theoretic approach to martingale measures in incomplete markets. *Surveys of Applied and Industrial Mathematics* 15, 18-24 (2008).
- [11] H. PHÂM : Optimal Stopping of Controlled Jump diffusion processes. A viscosity solution approach. *Journal of Mathematical Systems, Estimations and control*, vol 8 n°1, 1-27 (1998).